Worcester County Mathematics League

Varsity Meet 1 - October 26, 2022

COACHES' COPY
ROUNDS, ANSWERS, AND SOLUTIONS

Worcester County Mathematics League
Varsity Meet 1 - October 26, 2022
Answer Key

Round 1 - Arithmetic

1. $\dfrac{21}{13}$ or $1\dfrac{8}{13}$ or $1.\overline{615384}$
2. $\dfrac{4}{35}$ or $0.1\overline{142857}$
3. 5

Round 2 - Algebra I

1. $\dfrac{19}{5}$ or $3\dfrac{4}{5}$ or 3.8
2. 19
3. $\left(-\dfrac{1}{2}, -\dfrac{5}{2}\right)$ or $\left(-\dfrac{1}{2}, -2\dfrac{1}{2}\right)$ or $(-0.5, -2.5)$ (exact order)

Round 3 - Set Theory

1. $\{0, 4, 8, 12, 16, 20\}$ (any order, braces optional)
2. $A \cap (B \cup C)$ or $(B \cup C) \cap A$ or $A \cap (C \cup B)$ or $(C \cup B) \cap A$
3. 16

Round 4 - Measurement

1. $\{80, 20\}$ (either order, braces optional)
2. $(4, -20, 25, 0)$ (exact order only)
3. $(96, 16, 3, -32)$ (exact order only)

Round 5 - Polynomial Equations

1. -16
2. $(4, -1, -8)$ or $(-1, 4, -8)$ (either of these two orders)
3. 128

Team Round

1. $\dfrac{32}{9}$ or $3\dfrac{5}{9}$ or $3.\overline{5}$
2. 141
3. 5
4. 330π
5. -1
6. $-\dfrac{b^2}{4a}$ (must be negative)
7. 

8. $\left(\sqrt{2}, \sqrt{3}, 2\right)$ (exact order)
9. $\{2, -1+\dfrac{\sqrt{2}}{2}, -1-\dfrac{\sqrt{2}}{2}\}$ or $\{2, \dfrac{-2+\sqrt{2}}{2}, \dfrac{-2-\sqrt{2}}{2}\}$ (any order, with or without parentheses, and may also combine two roots using $\pm$.)

All answers must be in simplest exact form in the answer section.

NO CALCULATORS ALLOWED

1. Evaluate the expression:

$$\cfrac{7}{6-\cfrac{5}{4-\cfrac{3}{2+1}}}$$

2. Evaluate the expression:

$$\frac{10 \div 5 \cdot ((5-8)-(10-15))}{1+2\cdot(3+4\cdot 5-6)}$$

3. Define the operation $a \oplus b$, operating on the numbers $0, 1, 2, 3, 4, 5, 6$, as the remainder after $a+b$ is divided by 7. Thus, $5 \oplus 6 = 4$. Likewise, $a \otimes b$ is defined as the remainder after $a \cdot b$ is divided by 7, where $\otimes$ is performed before $\oplus$ when evaluating expressions. Also, the operation a^n is performed before $\otimes$ and is defined as applying $\otimes$ to n copies of a. For instance, $2^3 = 2 \otimes 2 \otimes 2$. Find:

$$(3 \oplus 6)^5 \oplus (2 \oplus 4)^{12}$$

<u>ANSWERS</u>

(1 pt) 1. ______________________

(2 pts) 2. ______________________

(3 pts) 3. ______________________

Worc. Academy, St. Johns, Quaboag

Worcester County Mathematics League
Varsity Meet 1 - October 26, 2022
Round 2 - Algebra I

All answers must be in simplest exact form in the answer section.
NO CALCULATORS ALLOWED

1. Solve for x:

$$x^2 - 13 = (5 - x)^2$$

2. Keith has at least \$5.00, Jean has 20¢ more than Keith, and Eli has \$2.30 more than Keith. Together Keith, Jean, and Eli have exactly d dollars. What is the smallest possible integer value of d?

3. Solve the following system for (x, y):

$$\frac{2}{x-y} + \frac{1}{x+y} = \frac{2}{3}$$
$$\frac{5}{x-y} + \frac{6}{x+y} = \frac{1}{2}$$

ANSWERS

(1 pt) 1. ____________________

(2 pts) 2. ____________________

(3 pts) 3. $(x, y) =$ ______________

Clinton, Tahanto, Doherty

Worcester County Mathematics League

Varsity Meet 1 - October 26, 2022

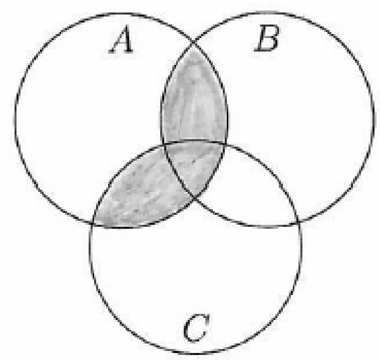
Round 3 - Set Theory

All answers must be in simplest exact form in the answer section.

NO CALCULATORS ALLOWED

1. Let U, the universal set, be $U = \{\text{all integers x such that } -10 \leq x \leq 21\}$, $A = \{\text{all possible absolute values of elements in } U\}$ and $B = \{\text{all multiples of 4 in } U\}$. Find the set $A \cap B$ and list its elements.

2. Write an expression involving intersections and/or unions of sets A, B, and C that correspond to the shaded area of the Venn Diagram shown below. A correct expression must have one instance each of A, B, and C and may include parentheses, but must *not* include the complement operation.



3. Ms. B. brought 81 students to an ice cream parlor for ice cream sundaes. Each student was given a choice of chocolate, vanilla, strawberry, or any combination of those three ice cream flavors. Each student chose at least one flavor; 40 chose vanilla, 36 chose chocolate, and 38 chose strawberry; 24 chose *only* vanilla; and 10 chose both vanilla and chocolate. Of the 10 who chose vanilla and chocolate, 7 also chose strawberry. How many students chose *only* chocolate?

ANSWERS

(1 pt) 1. $A \cap B = \{$_______________$\}$

(2 pts) 2. ____________________

(3 pts) 3. ____________________

Shrewsbury, QSC, Leicester

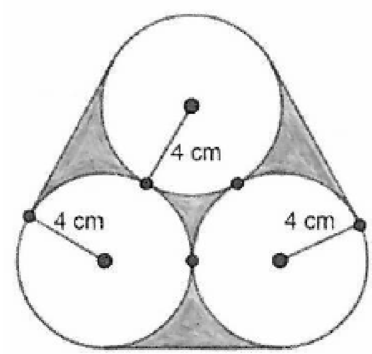
Worcester County Mathematics League
Varsity Meet 1 - October 26, 2022
Round 4 - Measurement

All answers must be in simplest exact form in the answer section.
NO CALCULATORS ALLOWED

1. The area of a rectangular field is 1600m^2 and its perimeter is 200m. Find the dimensions of the field (length X width). Order doesn't matter.

2. A square piece is cut out of each of the four corners of an aluminum sheet with dimensions 5cm X 5cm. Each square piece has side length x cm, where $x < 2.5$cm. The resulting four tabs are folded up to form an open rectangular box of height x. The volume of the box can be represented as a polynomial of the form $ax^3 + bx^2 + cx + d$. Find the ordered quadruple (a, b, c, d).

3. A rubber band is stretched around three congruent circles of radius 4cm, as shown in the figure to the right. Each of the circles is tangent to the other two. The area of the shaded region can be expressed in simplest radical form as $l + m\sqrt{n} + p\pi$ where l, m, n, p are integers. Find the ordered quadruple (l, m, n, p).



ANSWERS

(1 pt) 1. __________ m X __________ m

(2 pts) 2. $(a, b, c, d) = ($ ____________ $)$

(3 pts) 3. $(l, m, n, p) = ($ ____________ $)\text{cm}^2$

South, Shrewsbury, Worc. Academy/QSC

Worcester County Mathematics League

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Round 5 - Polynomial Equations

All answers must be in simplest exact form in the answer section.

NO CALCULATORS ALLOWED

1. One root of $6x^2 + bx - 6 = 0$ is 3. Find b.

2. The difference of the two roots (r_1 and r_2) of $2x^2 - 6x + c$ is 5. Find the ordered triple (r_1, r_2, c).

3. Let x be a real number such that $x^3 + 4x = 8$. Determine the value of $x^7 + 64x^2$.

ANSWERS

(1 pt) 1. $b =$ ____________________

(2 pts) 2. $(r_1, r_2, c) =$ (______________)

(3 pts) 3. __________________________

Bromfield, Tantasqua, Westborough

Worcester County Mathematics League
Varsity Meet 1 - October 26, 2022
Team Round

All answers must be in simplest exact form in the answer section.
NO CALCULATORS ALLOWED

1. Let $a \boxdot b = 2ab - b^2$ and $a \triangle b = \dfrac{a}{b} - a$. Evaluate $3 \boxdot (-1 \triangle 3)$.

2. If $\dfrac{42+43+44}{3} = \dfrac{2020+2021+2022}{n}$, find n.

3. A set containing $k+1$ elements has 48 more subsets than a set containing $k-1$ elements. Find k.

4. Find the exact surface area of the cylinder with a radius of 12cm and a height of $\dfrac{7}{4}$cm. Express your answer in terms of π.

5. Find all real solutions to the following equation in x:
$$\frac{x+2}{5x^2+9x+5} = \frac{2x-1}{5x^2+3x-5}$$

6. Astronaut Sally Ride entered a high jump competition on the moon. Her vertical jump was represented as a function of time by
$$h(t) = at^2 + bt$$
where $a < 0$ and $b > 0$. Express the maximum height of her jump in terms of a and b.

7. Graph the set described below on the real number line (on the Team Round answer sheet):
$$\{x : 3x + 12 \geq -6\} \cup \{x : -7x - 7 \leq -28\}$$

8. An archery target consists of four concentric circles with radii $r_1 < r_2 < r_3 < r_4$, with $r_1 = 1$ft. The three areas between circles (circle pairs 1 and 2, 2 and 3, and 3 and 4) are equal to each other and equal to the area of the smallest circle. Find the ordered triple (r_2, r_3, r_4), expressed in simplest radical form.

9. Find all three roots of the following polynomial equation:
$$2x^3 - 7x - 2 = 0$$

Worcester County Mathematics League
Varsity Meet 1 - October 26, 2022
Team Round Answer Sheet

ANSWERS

1. ____________________

2. ____________________

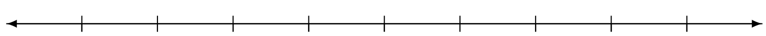
3. ____________________

4. ____________________ cm^2

5. ____________________

6. ____________________

7.



8. $(r_2, r_3, r_4) = ($______________$)$ ft

9. $\{$ ____________________ $\}$

Algonq., Burncoat/QSC, Algonq., Clinton, AMSA, Worcester Acad., Milbury, Shepherd Hill, Notre Dame

Worcester County Mathematics League
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2. 19
3. $\left(-\frac{1}{2}, -\frac{5}{2}\right)$ or $\left(-\frac{1}{2}, -2\frac{1}{2}\right)$ or $(-0.5, -2.5)$
(exact order)

Round 3 - Set Theory

1. $\{0, 4, 8, 12, 16, 20\}$ (any order, braces optional)
2. $A \cap (B \cup C)$ or $(B \cup C) \cap A$
or $A \cap (C \cup B)$ or $(C \cup B) \cap A$
3. 16

Round 4 - Measurement

1. $\{80, 20\}$ (either order, braces optional)
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(any order, with or without parentheses, and may also combine two roots using $\pm$.)

Round 1 - Arithmetic

1. Evaluate the expression:

$$\cfrac{7}{6-\cfrac{5}{4-\cfrac{3}{2+1}}}$$

Solution: Evaluate the expression starting from the bottom right fraction and working left and up: $\frac{3}{2+1}=\frac{3}{3}=1$. Then $4-\frac{3}{2+1}=4-1=3$, $\cfrac{5}{4-\cfrac{3}{2+1}}=\frac{5}{3}$, $6-\cfrac{5}{4-\cfrac{3}{2+1}}=6-\frac{5}{3}=\frac{18-5}{3}=\frac{13}{3}$, and $\cfrac{7}{6-\cfrac{5}{4-\cfrac{3}{2+1}}}=\cfrac{7}{\frac{13}{3}}=7\cdot\frac{3}{13}=\boxed{\frac{21}{13}}$.

For the last step, the reciprocal rule was applied: $a\div\frac{b}{c}=a\cdot\frac{c}{b}$.

2. Evaluate the expression:

$$\frac{10\div 5\cdot((5-8)-(10-15))}{1+2\cdot(3+4\cdot 5-6)}$$

Solution: We will evaluate the numerator (n) and denominator (d) of this rational expression separately using order of operations. First, evaluate the expression in the innermost parentheses in the numerator, then in the outer parentheses noting that subtracting a negative is the same as adding a positive: $n=10\div 5\cdot((5-8)-(10-15))=10\div 5\cdot(-3-(-5))=10\div 5\cdot 2=$. Next, evaluate left to right, since division and multiplication have the same priority: $n=(10\div 5)\cdot 2=2\cdot 2=4$.

Now evaluate the denominator starting with the expression in the parentheses and noting that multiplication is performed before addition or subtraction: $d=1+2\cdot(3+4\cdot 5-6)=1+2\cdot(3+(4\cdot 5)-6)=1+2\cdot(3+20-6)=1+2\cdot 17=1+34=35$.

Finally, the answer is $\frac{n}{d}=\boxed{\frac{4}{35}}$, since the numerator and denominator have no common factors to cancel.

3. Define the operation $a \oplus b$, operating on the numbers $0, 1, 2, 3, 4, 5, 6$, as the remainder after $a + b$ is divided by 7. Thus, $5 \oplus 6 = 4$. Likewise, $a \otimes b$ is defined as the remainder after $a \cdot b$ is divided by 7. Also, the operation a^n is defined as applying $\otimes$ to n copies of a. For instance, $2^3 = 2 \otimes 2 \otimes 2$. The usual order of operations applies, with $\oplus$ taking the place of addition and $\otimes$ taking the place of multiplication. Find:

$$(3 \oplus 6)^5 \oplus (2 \oplus 4)^{12}$$

Solution: First evaluate the operations inside the parentheses: $3 \oplus 6 = 2$ and $2 \oplus 4 = 6$. Then $(3 \oplus 6)^5 = 2 \otimes 2 \otimes 2 \otimes 2 \otimes 2 = \text{rem}(2^5, 7) = \text{rem}(32, 7)$, where $\text{rem}(a, b)$ is the remainder after a is divided by b. Thus, $\text{rem}(32, 7) = 4 = (3 \oplus 6)^5$.

Next, $(2 \oplus 4)^{12} = 6^{12}$. Rather than multiplying by 6 eleven times, it is simpler to first calculate $6^2 = 6 \otimes 6 = \text{rem}(36, 7) = 1$, and note that $6^{12} = (6^2)^6$, so that $(2 \oplus 4)^{12} = (6^2)^6 = 1^6 = 1$ because $1 \otimes 1 = 1$. Putting the two results together, $(3 \oplus 6)^5 \oplus (2 \oplus 4)^{12} = 4 \oplus 1 = \boxed{5}$.

Round 2 - Algebra I

1. Solve for x:

$$x^2 - 13 = (5 - x)^2$$

Solution: First, note the identity $(a - b)^2 = a^2 - 2ab + b^2$, so the right hand side evaluates to $(5-x)^2 = 25-10x+x^2$. Then the solution to the equation of two quadratics can be shown in column form:

$$\begin{aligned} x^2 - 13 &= x^2 - 10x + 25 \\ 10x - x^2 + 13 &= -x^2 + 10x + 13 \end{aligned}$$

$$10x = 25 + 13 = 38$$

collecting terms with x on one side of the equation. Note that the x^2 term conveniently cancels, and $x = \frac{38}{10} = \boxed{\frac{19}{5}}$.

2. Keith has at least \$5.00, Jean has 20¢ more than Keith, and Eli has \$2.30 more than Keith. Together Keith, Jean, and Eli have exactly d dollars. What is the smallest possible integer value of d?

Solution: First define three variables: k = Keith's money in dollars, j = Jean's money in dollars, and e = Eli's money in dollars. Then the given information can be written as (dropping the dollar signs for clarity):

$$\begin{aligned} k &\geq 5.00 \\ j &= k + 0.20 \\ e &= k + 2.30 \\ k + j + e &= d \end{aligned}$$

Next, substitute the expressions for j and e into the last equation: $k+k+0.20+k+2.30 = 3k+2.50 = d$. Note that d is an integer, so $3k+2.50$ must be an integer. The smallest value of $3k$ such that $3k+2.50$ is an integer and $k \geq 5$ (so that $3k \geq 15$) is $3k = 16.50$ because 3 does not divide 15.50. Therefore $d = 3k + 2.50 = 16.50 + 2.50 = \boxed{19}$ dollars.

3. Solve the following system for (x, y):

$$\frac{2}{x-y} + \frac{1}{x+y} = \frac{2}{3}$$
$$\frac{5}{x-y} + \frac{6}{x+y} = \frac{1}{2}$$

Solution: The given system can be converted into a linear system using an appropriate variable substitution. Let $u = \dfrac{1}{x-y}$ and $v = \dfrac{1}{x+y}$. Then:

$$2u + v = \frac{2}{3}$$
$$5u + 6v = \frac{1}{2}$$

Multiply the top equation by 6 and subtract the bottom equation to eliminate v:

$$12u + 6u = 4$$
$$5u + 6u = \frac{1}{2}$$

$$7u \qquad = \frac{7}{2}$$

Thus $u = \dfrac{1}{2}$. Plug this value into the first equation: $2\left(\dfrac{1}{2}\right) + v = \dfrac{2}{3}$, so $v + 1 = \dfrac{2}{3}$ and $v = \dfrac{2}{3} - 1 = -\dfrac{1}{3}$. Recall that u and v are defined in terms of x and y, so that $\dfrac{1}{2} = \dfrac{1}{x-y}$ and $-\dfrac{1}{3} = \dfrac{1}{x+y}$. Cross multiply these two equations to make a linear system in x and y:

$$x - y = 2$$
$$x + y = -3$$

Add these two equations to eliminate y, leaving $2x = 2 - 3 = -1$, and $x = -\dfrac{1}{2}$. Plug this value for x into the second equation: $-\dfrac{1}{2} + y = -3$, or $y = -\dfrac{5}{2}$. In summary, $(x, y) = \boxed{\left(-\frac{1}{2}. -\frac{5}{2}\right)}$.

Round 3 - Set Theory

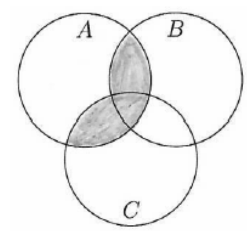
1. Let U, the universal set, be $U = \{$all integers x such that $-10 \leq x \leq 21\}$, $A = \{$all possible absolute values of elements in $U\}$ and $B = \{$all multiples of 4 in $U\}$. Find the set $A \cap B$ and list its elements.

Solution: Since U is the Universal Set, A and B must be subsets of U. Thus:

$$A = \{|-10|, |-9|, ..., |-1|, |0|, |1|, ..., |20|, |21|\} = \{0, 1, ..., 20, 21\}$$
$$B = \{-8, -4, 0, 4, 8, 12, 16, 20\}$$

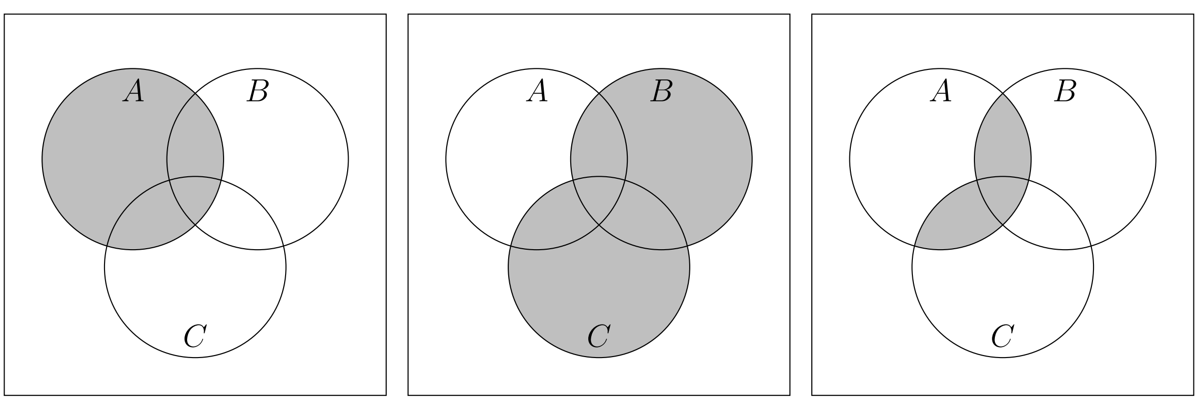
And $A \cap B = \boxed{\{0, 4, 8, 12, 16, 20\}}$. (The intersection consists of all positive elements of U that are multiples of 4).

2. Write an expression involving intersections and/or unions of sets A, B, and C that correspond to the shaded area of the Venn Diagram shown at right. A correct expression must have one instance each of A, B, and C and may include parentheses, but must *not* include the complement operation.



Solution: Note that the entire shaded area lies within circle A, and it also lies within the combined areas of B and C. Therefore all members of the set represented by the shaded area are in A *and* they are also in B *or* C. This observation suggests the expression $A \cap (B \cup C)$, because the union of two sets is all elements in either one set *or* the other and the intersection of two sets is all elements in one set *and* the other.

This expression can be checked using Venn diagrams, as shown below. The shaded area in the Venn diagram on the left below is set A. The shaded area on the middle diagram is $B \cup C$. The shaded area in the diagram on the right is the common area between these two shadings, so it represents $\boxed{A \cap (B \cup C)}$, which is the description of the Venn Diagram shading in the problem statement.



3. Ms. B. brought 81 students to an ice cream parlor for ice cream sundaes. Each student was given a choice of chocolate, vanilla, strawberry, or any combination of those three ice cream flavors. Each student chose at least one flavor; 40 chose vanilla, 36 chose chocolate, and 38 chose strawberry; 24 chose *only* vanilla; and 10 chose both vanilla and chocolate. Of the 10 who chose vanilla and chocolate, 7 also chose strawberry. How many students chose *only* chocolate?

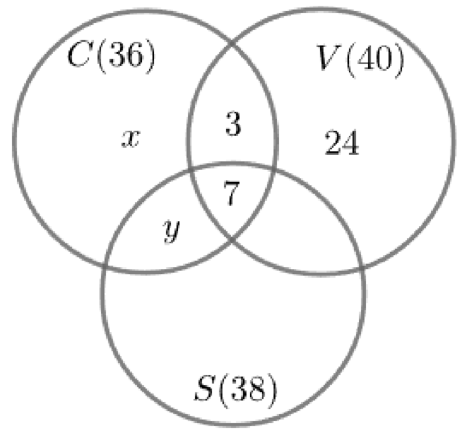
Solution: This problem can be solved by applying the given information to a Venn Diagram, one fact at a time. Define sets as follows: $C = \{\text{students who chose chocolate}\}$, $V = \{\text{students who chose vanilla}\}$, and $S = \{\text{students who chose strawberry}\}$. Also, let $\overline{A}$ denote the complement of set A and $|A|$ denote the number of elements in set A.

The diagram at right is labeled with the number of elements in some of the regions. Thus, 7 students chose chocolate, vanilla, and strawberry so $|C \cap V \cap S| = 7$ (the center region), 24 students only chose vanilla so $|V \cap \overline{C} \cap \overline{S}| = 24$ (the upper right region), and ten students chose chocolate and vanilla so $|C \cap V| = 10$ (the two upper regions in the center sum to 10, so the top central region has $10 - 7 = 3$ elements). Let x be the number of students who only chose chocolate, which is the number to be found, so $x = |C \cap \overline{V} \cap \overline{S}|$ (the top left region has x elements).

Note that x can be found if $y = |C \cap S \cap \overline{V}|$ is known because then x would be the only unknown region in C. That is, $|C| = 36 = x + |C \cap V| + y$, so $x = 36 - |C \cap V| - y = 36 - 10 - y = 26 - y$ because $|C \cap V| = 10$ is given information.

The most direct method to find y is to apply the principle of inclusion-exclusion to find $|C \cap S|$: $|C \cup S| = |C| + |S| - |C \cap S| = 36 + 38 - |C \cap S| = 74 - |C \cap S|$. The total number of students is known ($|C \cup V \cup S| = 81$) and the number of students who only chose Vanilla is known ($|V \cap \overline{C} \cap \overline{S}| = 24$), so $|C \cup S| = |C \cup V \cup S| - |V \cap \overline{C} \cap \overline{S}| = 81 - 24 = 57$. Now $|C \cap S| = 74 - |C \cup S| = 74 - 57 = 17$.

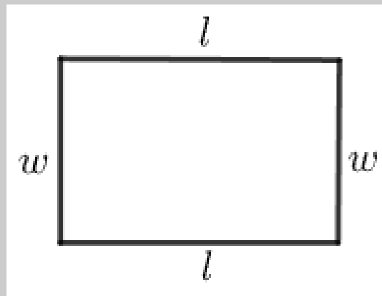
Next, $|C \cap S| = 17 = y + |C \cap V \cap S| = y + 7$, so $y = 10$. So $x = 26 - y = 26 - 10 = \boxed{16}$.



Round 4 - Measurement

1. The area of a rectangular field is 1600m^2 and its perimeter is 200m. Find the dimensions of the field (length X width). Order doesn't matter.

Solution: Let l = length of the floor (in feet) and w = width of the floor (in feet) as shown in the figure below.



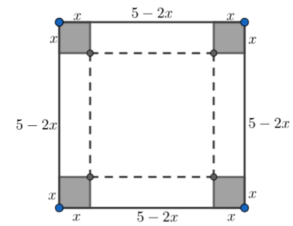
Then the area is $l \cdot w = 1600$ and the perimeter is $2(l + w) = 200$. Therefore $l + w = 100$. If l and w are integers, one method is to search through all pairs of possible numbers whose product is 1600 and select the pair whose sum is 100. veNote that the area has a factor of 100. or $10 \cdot 10$. Since l and w are positive, they are both less than 100, the perimeter, which restricts the search:

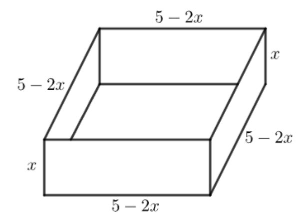
$$\begin{aligned} 1600 &= 50 \cdot 32 \\ &= 40 \cdot 40 \\ &= 25 \cdot 32 \\ &= 20 \cdot 80 \end{aligned}$$

All other factorings have one dimension of 100 or greater. Clearly, only $20 + 80 = 100$, so $\{l, w\} = \boxed{\{80, 20\}}$ (or the reverse order).

2. A square piece is cut out of each of the four corners of an aluminum sheet with dimensions 5cm X 5cm. Each square piece has side length x cm, where $x < 2.5$cm. The resulting four tabs are folded up to form an open rectangular box of height x. The volume of the box can be represented as a polynomial of the form $ax^3 + bx^2 + cx + d$. Find the ordered quadruple (a, b, c, d).

Solution: The cutting and folding process is shown in the diagram to the right. A square (shaded) of side length $x$cm is cut out each corner, and the sheet is folded in four places along the dotted lines, each fold at right angle to the square base. The folding creates a rectangular box of height $x$cm and with a square base of side length $5 - 2x$cm, as shown in the bottom figure.
The volume of the box is the product of the area of its base and its height. Expand the expression for the area of the base $(5 - 2x)^2$ using the identity $(a+b)^2 = a^2 + 2ab + b^2$ and distribute the height (x) over the resulting polynomial to put the volume into the desired polynomial form:

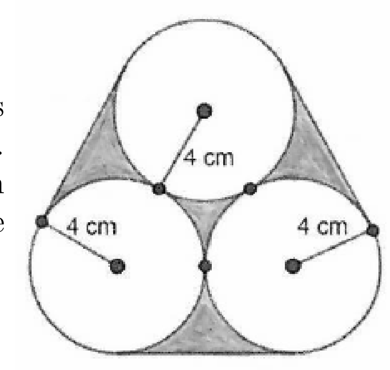




$$
\begin{aligned}
V &= (5 - 2x)^2 \cdot x \\
&= \left(4x^2 - 20x + 25\right) \cdot x \\
&= 4x^3 - 20x^2 + 25x + 0
\end{aligned}
$$

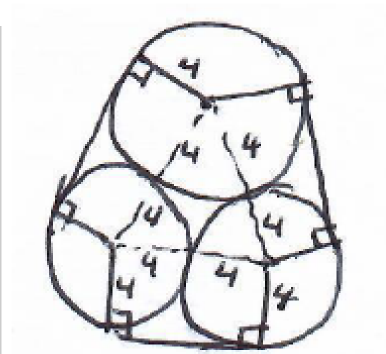
Thus, $(a, b, c, d) = \boxed{(4, -20, 25, 0)}$

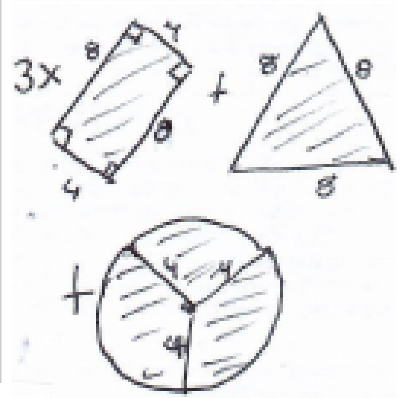
3. A rubber band is stretched around three congruent circles of radius 4cm, as shown in the figure below. Each of the circles is tangent to the other two. The area of the shaded region can be expressed in simplest radical form as $l + m\sqrt{n} + p\pi$ where l, m, n, p are integers. Find the ordered quadruple (l, m, n, p).L



Solution: The shaded area can be found by subtracting the areas of the three circles from the entire area enclosed by the rubber band. Call that area A_1. To find A_1, first draw two radii for each circle to the points of tangency of the rubber band with the circle, as shown in the top figure to the right. Note that the radii are perpendicular to the straight line of the rubber band. Then draw two radii for each circle to the points of tangency with the other two circles. Note that two radii drawn to the same point of tangency are collinear so that they form a line segment of length twice the radius ($8cm$).

Note that the 12 radii divide the area A_1 into three ($4cm$X$8cm$) rectangles, a central triangle, and three sectors as can be seen in the top figure. A_1 can be calculated by summing the areas of these seven pieces, as shown in the bottom figure. Note that the central triangle is an equalateral triangle with side lenght 8cm. Also, the three sectors are congruent and have a 120 deg central angle. The central angle of the sector together with two right angles from the rectangles and the 60 deg angle of the central triangle form a complete circle. That is, if y is the measure of the central angle, then $y + 90 + 90 + 60 = 360$ and $y = 120$. Therefore the three sectors combine to form a circle with radius 4cm.





Solution: (continued) Now the area of one of the rectangles is $4 \cdot 8 = 32\text{cm}^2$. The area of an equilateral triangle with side length s is $s^2\frac{\sqrt{3}}{4}$, so the area of this triangle is $\frac{8^2\sqrt{3}}{4} = 16\sqrt{3}$, and the area of the 4cm radius circle is $\pi 4^2 = 16\pi$. So $A_1 = 3(\text{rectangle area}) + (\text{triangle area}) + (\text{area of 4cm circle}) = 3(32) + 16\sqrt{3} + 16\pi\ \text{cm}^2$.

Finally, we need to subtract the area of the three circles to find the shaded area: $A_1 - 3\pi(4^2) = A_1 - \pi \cdot 3 \cdot 16 = A_1 - 48\pi = 96 + 16\sqrt{3} + 16\pi - 48\pi = 96 + 16\sqrt{3} - 32\pi$ and $(l, m, n, p) = \boxed{(96, 16, 3, -32)}$.

Round 5 - Polynomial Equations

1. One root of $6x^2 + bx - 6 = 0$ is 3. Find b.

Solution: There are multiple ways to solve for b. The easiest way is to substitute 3 for x, simplify that equation, and then solve for b:b

$$\begin{aligned} 6(3)^2 + 3b - 6 &= 0 \\ 54 + 3b - 6 &= 0 \\ 48 + 3b &= 0 \\ 3b &= -48 \end{aligned}$$

Therefore, $b = \boxed{-16}$.

2. The difference of the two roots of $2x^2 - 6x + c$ is $r_1 - r_2 = 5$. Find the ordered triple (r_1, r_2, c).

Solution: First, if r_1 and r_2 are roots of a quadratic, then the quadratic can be written as $a(x - r_1)(x - r_2) = a\left(x^2 - (r_1 + r_2)x + r_1 r_2\right)$. The first step is to divide the polynomial by 2 (because $a = 2$) so that the constant term is the product of the roots and the coefficient that multiples the x term is the negative of the sum (of the roots:

$$2x^2 - -6x + c = 2\left(x^2 - 3x + \frac{c}{2}\right) = 2\left(x^2 - (r_1 + r_2)x + r_1 r_2\right)$$

Equating the constant terms from the last two equations above yields $\frac{c}{2} = r_1 \cdot r_2$, and equating the x coefficients results in $r_1 + r_2 = 3$. Also, it was given that $r_1 - r_2 = 5$. Add these two equations to eliminate r_2:

$$\begin{aligned} r_1 + r_2 &= 3 \\ r_1 - r_2 &= 5 \\ \hline 2r_1 &= 8 \end{aligned}$$

Thus, $r_1 = \frac{8}{2} = 4$. Substitute $r_1 = 4$ in the top equation and $4 + r_2 = 3$, so $r_2 = 3 - 4 = -1$. Finally, as noted above, $\frac{c}{2} = r_1 \cdot r_2$ so $c = 2 \cdot r_1 \cdot r_2 = 2 \cdot 4 \cdot (-1) = -8$. Thus, $(r_1, r_2, c) = \boxed{(4, -1, -8)}$.

3. Let x be a real number such that $x^3 + 4x = 8$. Determine the value of $x^7 + 64x^2$.

Solution: The degree of the polynomial $x^7 + 64x^2$ can be reduced by rewriting the given equation as $x^3 = 8 - 4x$ and substituting $8 - 4x$ for x^3. Proceeding with this idea, $x^6 = (x^3)^2 = (8 - 4x)^2 = 16x^2 - 64x + 64$, using the identity $(a - b)^2 = a^2 - 2ab + b^2$. Next,

$$x^7 = x \cdot x^6 = x\left(16x^2 - 64x + 64\right) = 16x^3 - 64x^2 + 64x$$

Now substitute $x^3 = 8 - 4x$ once again, and

$$x^7 = 16\,(8 - 4x) - 64x^2 + 64x = 128 - 64x - 64x^2 + 64x = 128 - 64x^2$$

. We are approaching the end of the tunnel. Substitute this expression for x^7 into the polynomial, and $x^7 + 64x^2 = 128 - 64x^2 + 64x^2 = \boxed{128}$.

Alternate solution: The answer may be found more directly using polynomial division. Let $P(x) = x^7 + 64x^2$ and $D(x) = x^3 + 4x - 8$. Note that $D(x) = 0$ from the given information. When $P(x)$ is divided by $D(x)$ there willl be a quotient and a remainder:

$$\frac{P(x)}{D(x)} = Q(x) + \frac{R(x)}{D(x)}$$

where $Q(x)$ is the quotient polynomial and $R(x)$ is the remainder polynomial resulting from the polynomial division. Multiply both sides of the equation by $D(x)$ to yield:

$$P(x) = Q(x)D(x) + R(x)$$

and we find that $P(x) = R(x)$ when $D(x) = 0$. Now divide $P(x)$ by $D(x)$ to find $R(x)$:

$$
\begin{array}{r|rrrrrrr}
 & & & x^4 & & -4x^2 & +8x & +16 \\ \cline{2-8}
x^3 + 4x - 8 & x^7 & & & & +64x^2 & & \\
 & -x^7 & -4x^5 & +8x^4 & & & & \\ \cline{2-4}
 & & -4x^5 & +8x^4 & & +64x^2 & & \\
 & & 4x^5 & & +16x^3 & -32x^2 & & \\ \cline{3-6}
 & & & 8x^4 & +16x^3 & +32x^2 & & \\
 & & & -8x^4 & & -32x^2 & +64x & \\ \cline{4-7}
 & & & & 16x^3 & & +64x & \\
 & & & & -16x^3 & & -64x & +128 \\ \cline{5-8}
 & & & & & & & 128
\end{array}
$$

So that $R(x) = 128$, and therefore $P(x) = \boxed{128}$ when $x^3 + 4x = 8$.

Team Round

1. Let $a \boxdot b = 2ab - b^2$ and $a \bigtriangleup b = \dfrac{a}{b} - a$. Evaluate $3 \boxdot (-1 \bigtriangleup 3)$.

Solution: Start by evaluating the operation within the parentheses: $-1 \bigtriangleup 3 = \dfrac{-1}{3} - (-1) = \dfrac{-1}{3} + 1 = \dfrac{2}{3}$. Now, substitute this value into the expression and evaluate the second operation: $3 \boxdot (-1 \bigtriangleup 3) = 3 \boxdot \dfrac{2}{3} = 2(3)\left(\dfrac{2}{3}\right) - \left(\dfrac{2}{3}\right)^2 = 4 - \dfrac{4}{9} = \dfrac{36-4}{9} = \boxed{\dfrac{32}{9}}$.

2. If $\dfrac{42+43+44}{3} = \dfrac{2020+2021+2022}{n}$, find n.

Solution: First, observe that dividing the sum of three numbers by 3 yields the average of the three numbers, and that the average of three consecutive integers is the middle number. Therefore $\dfrac{42+43+44}{3} = 43$. Likewise, the sum of three consecutive integers is three times the middle number, so $\dfrac{2020+2021+2022}{n} = \dfrac{3 \cdot 2021}{n}$. Thus, the original equation simplifies to $43 = \dfrac{3 \cdot 2021}{n}$, or

$$n = \frac{3 \cdot 2021}{43}$$

Dividing 2021 by 43, we find that $2021 = 43 \cdot 47$. Therefore $n = \dfrac{3 \cdot 47 \cdot 43}{43} = 3 \cdot 47 = \boxed{141}$.

3. A set containing $k+1$ elements has 48 more subsets than a set containing $k-1$ elements. Find k.

Solution: Note that a set with k elements has 2^k subsets (there are two possibilities for each element, present in the subset or not present in the subset; the choice of presence is independent for each of the elements, so there are $2 \cdot 2 \cdot ... \cdot 2$ different subsets possible, where there are n 2's in the product). Applying this result to the given information yields the equation:

$$2^{(k+1)} - 2^{(k-1)} = 48$$

Next, factor out $2^{(k-1)}$ from the two left hand side terms, simplify, and divide both sides by 3:

$$\begin{aligned} 2^{(k-1)}\left(2^2 - 2^0\right) &= 48 \\ 2^{(k-1)}(4-1) &= 48 \\ 2^{(k-1)} \cdot 3 &= 48 \\ 2^{(k-1)} &= 16 \end{aligned}$$

Finally, $16 = 2^4 = 2^{(5-1)}$, $2^{(k-1)} = 2^{(5-1)}$, and $k = \boxed{5}$ after equating the two exponents.

4. Find the exact surface area of the right cylinder with a radius of 12cm and a height of $\frac{7}{4}$cm. Express your answer in terms of π.

Solution: Let r be the radius of the two bases (congruent circles) and h be the height of the cylinder. The surface area (SA) of a right cylinder is twice the base area ($2\pi r^2$) plus the lateral area ($2\pi r \cdot h$). Therefore $SA = 2\pi r\,(r+h)$, where $2\pi r$ is factored out of both terms. Now plug $r = 12$ and $h = \frac{7}{4}$ into this expression:

$$\begin{aligned} SA &= 2\pi \cdot 12 \left(12 + \frac{7}{4}\right) \\ &= \pi \cdot 24 \left(\frac{48+7}{4}\right) \\ &= \pi \left(24 \cdot \frac{55}{4}\right) \\ &= \pi\,(6 \cdot 55) \end{aligned}$$

and the surface area is equal to $\boxed{330\pi}$.

5. Find all real solutions to the following equation:

$$\frac{x+2}{5x^2+9x+5} = \frac{2x-1}{5x^2+3x-5}$$

Solution: To solve, first cross multiply the equation (multiply both sides of the equation by the product of the two denominators, turning each rational expression into a product of two polynomials). Then multiply the polynomials on both sides and collect terms. Finally, combine all terms on one side to create a simple polynomial equation:

$$\begin{aligned} (x+2)\left(5x^2+3x-5\right) &= (2x-1)\left(5x^2+9x+5\right) \\ 5x^3+3x^2-5x+10x^2+6x-10 &= 10x^3+18x^2+10x-5x^2-9x-5 \\ 5x^3+13x^2+x-10 &= 10x^3+13x^2+x-5 \\ 0 = (10-5)x^3+(13-13)x^2+x-x+-5+10 \\ 0 &= 5x^3+5 \end{aligned}$$

Now divide by 5 to yield $x^3+1=0$, or $x^3=-1$, and $x = \boxed{-1}$ because there is only one real cube root of -1.

6. Astronaut Sally Ride entered a high jump competition on the moon. Her vertical jump was represented as a function of time by

$$h(t) = at^2 + bt$$

where $a < 0$ and $b > 0$. Express the maximum height of her jump in terms of a and b.

Solution: Note that $h(t)$ is a quadratic, and the graph of a quadratic is a parabola. Since $a < 0$, the parabola opens down. The x-intercepts of the parabola are the two solutions of $h(t) = 0 = at^2 + bt = t\,(at + b)$, or $x = 0$ and $x = \frac{-b}{a}$, which follows from the factored polynomial. Now, the maximum value of $h(t)$ occurs at the vertex of the parabola. The x-coordinate of the vertex lies on the parabola's axis of symmetry, which bisects the segment joining the two x-intercepts. Thus, the x-coordinate of the axis, as well as the x-coordinate of the vertex, is equal to $\frac{1}{2}\left(\frac{-b}{a} - 0\right) = -\frac{b}{2a}$. The maximum height (y-coordinate of the vertex) is found by evaluating $h(-\frac{b}{2a})$:

$$\begin{aligned} h(-\frac{b}{2a}) &= a\left(-\frac{b}{2a}\right)^2 + b\left(-\frac{b}{2a}\right) = a\frac{b^2}{4a^2} - \frac{b^2}{2a} \\ &= \frac{b^2}{4a} - \frac{b^2}{2a} = \boxed{-\frac{b^2}{4a}} \end{aligned}$$

This answer looks wrong, but recall that $a < 0$ and $b > 0$, so that this expression is always a positive number.

7. Graph the set described below on the real number line:

$$\{x : 3x + 12 \geq -6\} \cup \{x : -7x - 7 \leq -28\}$$

Solution: First solve the two inequalitites for x and not the the union of the two sets is the set of point that satisfy one or both of the two inequalities. Thus,

$$\begin{aligned} 3x + 12 &\geq -6 &\quad \text{or} \quad & -7x - 7 \leq -28 \\ x + 4 &\geq -2 &\quad \text{or} \quad & -x - 1 \leq -4 \\ x &\geq -6 &\quad \text{or} \quad & 3 \leq x \end{aligned}$$

where a common factor was divided out of both sides of the inequality in the second step. In the third step, x was isolated by adding an appropriate expression to both sides of the inequality. The last inequality is equivalent to $x \geq 3$, and the logical combination of $x \geq -6$ or $x \geq 3$ is $\boxed{x \geq -6}$.

8. An archery target consists of four concentric circles (numbered from the center 1,2,3, and 4) with radii $r_1 < r_2 < r_3 < r_4$ and with $r_1 = 1$ft. The three areas between circles (circle pairs 1 and 2, 2 and 3, and 3 and 4) are equal to each other and equal to the area of the smallest circle. Find the ordered triple (r_2, r_3, r_4), expressed in simplest radical form.

Solution: The area of circle 1 is $\pi(1)^2 = \pi$. The between two concentric circles is equal to the area of the outer circle minus the area of the inner circle. Thus, the area between circles 2 and 1 is equal to $\pi r_2^2 - \pi r_1^2 = \pi r_2^2 - \pi = \pi r_1^2 = \pi$. Therefore $\pi r_2^2 = 2\pi$, or $r_2^2 = 2$. Similarly, $r_3^2 = r_2^2 + 1 = 3$, and $r_4^2 = r_3^2 + 1 = 4$. Taking the square roots of the radii results in $(r_2, r_3, r_4) = \boxed{(\sqrt{2}, \sqrt{3}, 2)}$, where the largest radius value is simplified by replacing $\sqrt{4}$ with 2.

9. Find all three roots of the following polynomial equation:

$$2x^3 - 7x - 2 = 0$$

Solution: If this cubic equation has a rational root, the quickest way to solve it is to find a rational root by trial and error and then find the depressed equation using synthetic division. The depressed equation will be a qudratic equation that can be solved by applying the quadratic formula if it can't be factored.

Applying the rational roots theorem, the possible rational roots of $p(x) = 2x^3 - 7x - 2$ are $\left\{\pm 1, \pm 2, \pm \frac{1}{2}\right\}$. Trying $x = 1$ first, $p(1) = 2 - 7 - 2 \neq 0$. Trying $x = 2$ next, $p(2) = 2(2)^3 - 7(2) - 2 = 2 \cdot 8 - 14 - 2 = 16 - 14 - 2 = 0$ and 2 is a root. Because 2 is a root, synthetic division can be used to divide $p(x)$ by $(x - 2)$ to find the depressed quadratic equation:

$$\begin{array}{r|rrrr} 2 & 2 & 0 & -7 & -2 \\ \hline & & 4 & 8 & 2 \\ \hline & 2 & 4 & 1 & 0 \end{array}$$

Alternatively, polynomial division could be used to find that the resulting quadratic is $p(x) \div (x-2) = 2x^2 + 4x + 1$. Applying the quadratic formula,

$$x = \frac{-4 \pm \sqrt{4^2 - 4 \cdot 2 \cdot 1}}{2 \cdot 2} = \frac{-4 \pm \sqrt{16 - 8}}{4} = \frac{-4 \pm \sqrt{8}}{4} = \frac{-4 \pm 2\sqrt{2}}{4} = \frac{-2 \pm \sqrt{2}}{2}$$

Therefore the three roots are $\boxed{2, -1 + \frac{\sqrt{2}}{2}, -1 - \frac{\sqrt{2}}{2}}$.